No bounded-edge-bipartisation high-odd-girth three-colour inequality senders

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Abstract

We prove that bounded edge-bipartisation and sufficiently large odd-girth prevent two distinct nonadjacent vertices from being forced to receive different colours in every proper three-colouring. More precisely, if S is a finite simple graph with $\beta(S) \leq K$ and $\mathrm{oddgirth}(S) \geq 3K+5$, then every distinct nonadjacent pair $u, v \in V(S)$ is equal-colourable in some proper 3-colouring of S. Consequently there is no family of finite 3-colourable graphs S_L with distinct nonadjacent terminals u_L, v_L, bounded $\beta(S_L)$, and $\mathrm{oddgirth}(S_L) \to \infty$, such that every proper 3-colouring separates u_L and v_L.

Disclosure and status

This note is a draft. It has not been independently peer reviewed, and it makes no claim of priority over the existing literature. AI was used in generating the draft, together with human guidance and minor editing.

1 Notation

All graphs are finite and simple. For a graph G, let $\beta(G)$ denote the minimum number of edges whose deletion makes G bipartite. The odd-girth $\mathrm{oddgirth}(G)$ is the length of a shortest odd cycle in G, with $\mathrm{oddgirth}(G) = \infty$ if G is bipartite.

A proper 3-colouring means a map

$$c : V(G) \to \{1, 2, 3\}$$

such that adjacent vertices receive distinct colours.

2 The protected-side lemma

This is the only combinatorial lemma needed later. It is stated in a form that is independent of colourings.

Lemma 2.1 (Protected-side lemma)**.** *Let $K, h \in \mathbb{N}_0$. Let G be a graph with a partition*

$$V(G) = X \cup Y,$$

and let $F \subseteq E(G)$ satisfy $|F| \leq K$ such that $G - F$ is bipartite with sides X, Y. Assume every edge of F has both endpoints in X or both endpoints in Y.

Let C be an odd cycle in G. Let $Z, Q \subseteq X$, with $|Z| \leq h$, and suppose

$$V(C) \cap X \subseteq Q \cup Z.$$

Then either

$$|C| \le 2h + K + 2,$$

or there are two distinct vertices $q, q' \in Q \cap V(C)$ such that one of the two q-to-q' arcs of C has odd length at most

$$2h + K + 2.$$

Proof. Put

$$D_C := E(C) \cap F.$$

Then $|D_C| \le |F| \le K$.

First suppose $V(C) \cap X = \varnothing$. Then all vertices of C lie in Y. Since $G - F$ has no Y-Y edge, every edge of C belongs to F. Hence

$$|C| \le |F| \le K \le 2h + K + 2,$$

so the first alternative holds.

Now suppose $V(C) \cap X \neq \varnothing$. List the vertices of $V(C) \cap X$ in cyclic order around C as

$$x_1, x_2, \ldots, x_s,$$

with indices modulo s. For each i, let P_i be the x_i-to-x_{i+1} arc of C whose internal vertices contain no vertex of X; when $s = 1$, this means that P_1 is the whole cycle, viewed as the closed arc from x_1 back to itself. Thus every internal vertex of P_i, if any, lies in Y. Let

$$d_i := |E(P_i) \cap F|.$$

We claim that

$$|P_i| \le d_i + 2 \tag{1}$$

for every i, where $|P_i|$ denotes the number of edges of the arc.

Indeed, if $|P_i| = 1$, then P_i is a same-side X-X edge, hence belongs to F, so $d_i = 1$ and $|P_i| = 1 \le d_i + 2$. If $|P_i| \ge 2$, then the first and last edges of P_i are X-Y edges, while every internal edge, if present, is a Y-Y edge. Since $G - F$ is bipartite with sides X, Y, every internal Y-Y edge lies in F, while the first and last X-Y edges do not lie in F, because every edge of F is same-side. Therefore, in this case,

$$|P_i| = d_i + 2.$$

This proves (1). Therefore, summing (1) over i gives

$$|C| = \sum_{i=1}^{s} |P_i| \le \sum_{i=1}^{s} (d_i + 2) = 2s + \sum_{i=1}^{s} d_i \le 2s + K. \tag{2}$$

Let

$$m := |Q \cap V(C)|.$$

If $m \le 1$, then all but at most one of the X-vertices of C lie in Z. Hence

$$s \le h + 1.$$

By (2),

$$|C| \le 2(h + 1) + K = 2h + K + 2,$$

so the first alternative holds.

It remains to consider $m \ge 2$. List the vertices of $Q \cap V(C)$ in cyclic order around C as

$$q_1, q_2, \ldots, q_m,$$

with indices modulo m. For each j, let R_j be the q_j-to-q_{j+1} arc of C whose internal vertices contain no vertex of Q. These arcs partition $E(C)$.

Let z_j be the number of internal X-vertices on R_j, and let

$$r_j := |E(R_j) \cap F|.$$

Every internal X-vertex of R_j lies in Z, because $V(C) \cap X \subseteq Q \cup Z$ and R_j has no internal Q-vertex. Thus $z_j \leq h$.

The arc R_j is a concatenation of $z_j + 1$ of the arcs P_i. Applying (1) to those arcs gives

$$|R_j| \leq 2(z_j + 1) + r_j.$$

Since $z_j \leq h$ and $r_j \leq |F| \leq K$, we obtain

$$|R_j| \leq 2(h + 1) + K = 2h + K + 2. \tag{3}$$

The arcs $R_1, \ldots, R_m$ partition the odd cycle C, so

$$|C| = \sum_{j=1}^{m} |R_j|$$

is odd. Therefore at least one arc R_j has odd length. By (3), that odd arc has length at most $2h + K + 2$, and its endpoints are two distinct vertices of $Q \cap V(C)$. This proves the second alternative. □

3 The equality-colouring criterion

We recall the exact criterion behind the terminal argument.

Lemma 3.1 (Equality-colouring criterion)**.** *Let S be a graph with distinct nonadjacent vertices u, v. Put*

$$M := S - \{u, v\}, \qquad N := \big(N_S(u) \cup N_S(v)\big) \cap V(M).$$

Then S has a proper 3-colouring with $c(u) = c(v)$ if and only if there exists an independent set

$$I \subseteq V(M) \setminus N$$

such that $M - I$ is bipartite.

Proof. If S has a proper 3-colouring with $c(u) = c(v)$, then after permuting the colour names we may assume $c(u) = c(v) = 1$. Let

$$I := \{x \in V(M) : c(x) = 1\}.$$

Then I is independent, no vertex of N lies in I, and $M - I$ is coloured with colours 2 and 3, hence is bipartite.

Conversely, suppose such an independent set I exists. Colour I with colour 1. Since $M - I$ is bipartite, colour its two sides with colours 2 and 3. Since $I \cap N = \varnothing$, no neighbour of u or v inside M receives colour 1. Colour both u and v with colour 1. Since u and v are nonadjacent, this is a proper 3-colouring of S with $c(u) = c(v)$. □

4 No bounded-β, high-odd-girth three-colour sender

Theorem 4.1. *Let $K \in \mathbb{N}_0$. Let S be a graph with*

$$\beta(S) \leq K$$

and

$$\mathrm{oddgirth}(S) \geq 3K + 5.$$

Then for every pair of distinct nonadjacent vertices $u, v \in V(S)$, there is a proper 3-colouring of S with

$$c(u) = c(v).$$

Proof. Choose $F_0 \subseteq E(S)$ with $|F_0| \leq K$ such that $S - F_0$ is bipartite. Let $X \cup Y$ be a bipartition of $S - F_0$.

If an edge $e \in F_0$ has one endpoint in X and the other in Y, then adding e back preserves bipartiteness with the same sides X, Y. Delete all such unnecessary cross-edges from F_0. Thus we obtain a set $F \subseteq F_0$ such that

$$|F| \leq K,$$

$S - F$ is bipartite with sides X, Y, and every edge of F has both endpoints in X or both endpoints in Y. Equivalently, every same-side edge of S with respect to $X \cup Y$ belongs to F.

Swap X and Y if necessary so that $u \in X$. Put

$$M := S - \{u, v\},$$

$$P := N_S(u) \cap V(M), \qquad Q := N_S(v) \cap V(M), \qquad N := P \cup Q,$$

and

$$X_M := X \cap V(M).$$

Let

$$E_W := \{e \in F : e \text{ has both endpoints in } X_M \setminus N\}.$$

For every edge of E_W, choose one endpoint and put it into a set W. Then

$$W \subseteq X_M \setminus N, \qquad |W| \leq |E_W|.$$

Define

$$I := (X_M \setminus N) \setminus W.$$

Then $I \subseteq V(M) \setminus N$.

We first check that I is independent. Since $I \subseteq X$, no edge of $S - F$ has both endpoints in I. Hence any edge of S with both endpoints in I would have to belong to F. But every edge of F with both endpoints in $X_M \setminus N$ was hit by W, and $I = (X_M \setminus N) \setminus W$. Thus no such edge has both endpoints in I, proving that I is independent.

It remains to show that $M - I$ is bipartite. Suppose not. Then $M - I$ contains an odd cycle C. Since I contains every vertex of $X_M \setminus N$ except those in W, every X-vertex of C lies in

$$(X \cap N) \cup W. \tag{4}$$

We consider two cases.

Case 1: $v \in X$**.** Let

$$A := X \cap N.$$

For each $x \in A$, choose a terminal defect edge $e_x \in F$ as follows. If $x \in P$, set $e_x := ux$; otherwise $x \in Q$, and we set $e_x := vx$. These edges are same-side edges, because $u, v, x \in X$, and hence

they belong to F. They are distinct as x varies over A. Moreover, none of them belongs to E_W, since every edge of E_W has both endpoints in $X_M \setminus N$, whereas each e_x is incident with u or v.

Thus

$$|A| + |E_W| \le |F| \le K.$$

Let

$$Z := A \cup W.$$

Then, by $|W| \le |E_W|$,

$$|Z| \le |A| + |W| \le |A| + |E_W| \le K.$$

By (4), every X-vertex of C lies in Z. Apply Lemma 2.1 with $Q = \varnothing$ and $h = K$. The second alternative is impossible because $Q = \varnothing$, so

$$|C| \le 2K + K + 2 = 3K + 2,$$

contradicting $\text{oddgirth}(S) \ge 3K + 5$.

Case 2: $v \in Y$**.** Let

$$Q_X := Q \cap X = N_S(v) \cap X \cap V(M).$$

Since u and v are nonadjacent and $v \in Y$, this is simply $N_S(v) \cap X$. These are the possibly numerous X-neighbours of v. Let

$$A := (X \cap N) \setminus Q_X.$$

If $x \in A$, then $x \in P$, because every vertex of $X \cap Q$ belongs to Q_X. Hence x is adjacent to $u \in X$, and the edge ux is a same-side edge. Therefore $ux \in F$.

The edges ux with $x \in A$ are distinct and are disjoint from E_W, since each of them is incident with u while every edge of E_W lies inside $X_M \setminus N$. Consequently

$$|A| + |E_W| \le |F| \le K.$$

Let

$$Z := A \cup W.$$

Then

$$|Z| \le |A| + |W| \le |A| + |E_W| \le K,$$

and by (4), every X-vertex of C lies in

$$Q_X \cup Z.$$

Apply Lemma 2.1 with $Q = Q_X$ and $h = K$. Either

$$|C| \le 2K + K + 2 = 3K + 2,$$

which contradicts $\text{oddgirth}(S) \ge 3K + 5$, or there are two distinct vertices $q, q' \in Q_X \cap V(C)$ joined along C by an odd arc of length at most $3K + 2$.

In the latter case, both q and q' are adjacent to v. Since $C \subseteq M$, the vertex v is not on C. Therefore the odd q-to-q' arc of C, together with the two edges vq and vq', forms an odd cycle in S of length at most

$$3K + 4.$$

This also contradicts $\text{oddgirth}(S) \ge 3K + 5$.

Both cases are impossible. Hence $M - I$ has no odd cycle. Therefore $M - I$ is bipartite.

We have constructed an independent set $I \subseteq V(M) \setminus N$ such that $M - I$ is bipartite. By Lemma 3.1, S has a proper 3-colouring with $c(u) = c(v)$. □

Remark 4.2. The constant $3K + 5$ is the direct output of the proof above. The proof does not claim that this numerical threshold is optimal.

Corollary 4.3. *There is no family of finite graphs S_L with distinct nonadjacent terminals u_L, v_L such that, for some constant B independent of L,*

$$\beta(S_L) \leq B, \qquad \operatorname{oddgirth}(S_L) \to \infty,$$

S_L has at least one proper 3-colouring, and every proper 3-colouring of S_L satisfies

$$c(u_L) \neq c(v_L).$$

In particular, the bounded-β, high-odd-girth $q = 3$ inequality-sender route is impossible.

Proof. If such a family existed, choose an integer $K \geq B$. Then $\beta(S_L) \leq K$ for every L. Since $\operatorname{oddgirth}(S_L) \to \infty$, choose L with $\operatorname{oddgirth}(S_L) \geq 3K + 5$. Theorem 4.1 gives a proper 3-colouring of S_L with $c(u_L) = c(v_L)$, contradicting the assumed inequality-sender property. □

5 Comments on the proof

The proof does not use a spare fourth colour. The mechanism is specific to three colours: bounded edge-bipartisation gives a bipartite base with at most K same-side defect edges. After fixing a bipartition side X containing one terminal, one deletes almost all non-terminal-neighbour vertices of X. This deleted set is made independent by sacrificing at most one endpoint from each X-X defect edge. Any odd cycle that survives must have all its X-vertices among terminal-neighbours and the sacrificed exceptional vertices. If the second terminal lies on the same bipartition side, there are only boundedly many such terminal-neighbours, and in fact these neighbours can be charged to defect edges disjoint from the sacrificed ones. If it lies on the opposite side, its possibly many X-neighbours are protected: two such neighbours lying close together along an odd arc immediately create a short odd cycle through the second terminal.